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Universality and Superuniversality of Multifractals in Nonlinear Resistor Networks

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The multifractal function $f(\alpha)$ is generalized to describe noisy nonlinear random resistor networks. An approximant function for the family of noise exponents is introduced that provides a good description of real percolative systems for strong nonlinearities. By mapping from this family to the multifractal function, one can approximate the latter. A scale transformation of α in the approximation makes the multifractal function universal for all nonlinearities and by applying an additional transformation, this function becomes superuniversal, i.e., independent of the dimension. The universality is demonstrated for the Mandelbrot–Given structure and the implications of these results are discussed on real percolative systems.

KEY WORDS: Nonlinear resistor networks; percolation model; noise exponents; multifractal function; universality; superuniversality.

1. INTRODUCTION

Multifractal analysis is a convenient tool to describe distributions whose moments scale independently of each other[(1–4)] at a critical point. Such a distribution appears within the framework of random resistor networks (RRNs) at the percolation threshold, when the elementary resistances are noisy.[(5–7)] The cumulants of the global resistance distribution $\langle R^q \rangle_c$ then scale as powers of the system size with an infinite set of independent exponents $\tilde{\psi}(q)$. This set is closely connected to the multifractal function $f(\alpha)$,[(7)] and there exists a one-to-one mapping between the two functions,[(7,8)] which shows that the information contained in $f(\alpha)$ already exists in $\tilde{\psi}(q)$.

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The study of noise and distributions extended to nonlinear RRNs whose elements obey[9,10]

$$V = r\,|I|^{\beta}\,\mathrm{Sign}(I) \tag{1}$$

where V, r, and I are the voltage, resistance, and current that a bond supports, respectively. Introducing noise to such systems at percolation, one obtains a generalized infinite set of exponents, $\tilde{\psi}(q, \beta)$.[8,10,11]

In this paper we study the universality properties of the noise exponents by generalizing the mapping between $\tilde{\psi}(q, \beta)$ and $f(\alpha, \beta)$. The main results are:

1. The generalization of the mapping between the nonlinear RRN noise exponents $\tilde{\psi}(q, \beta)$ and the multifractal function, which becomes β dependent, $f(\alpha, \beta)$.

2. The construction of an explicit approximant function (AF) for $f(\alpha, \beta)$ to describe actual percolative RRNs, and the study of its properties.

3. The AF becomes *universal* for all values of β when applying a scale transformation to α, thus implying an "almost" universality for real percolative networks.

4. The transformation of the AF to a *superuniversal* form, i.e., the function becomes independent of the dimension d as well as of β.

Finally, I discuss the experimental implications of the results and suggest explicit ways to check them.

2. NOISE IN NONLINEAR RRNs AND THE MAPPING BETWEEN $\tilde{\psi}(q, \beta)$ and $f(\alpha, \beta)$

Consider a nonlinear RRN that obeys (1), whose elementary resistors r are narrowly distributed around some mean value r_0 due to some physical mechanism. The cumulants of the global resistance distribution scale as powers of L, the separation between the terminals,

$$M_q(\beta) = \langle R^q \rangle_c \simeq \sum_{\text{bonds}} \left(\frac{\partial R}{\partial r_b}\right)^q \langle r^q \rangle_c \sim L^{\tilde{\psi}(q,\beta)} \tag{2}$$

These cumulants were shown to equal the $[(\beta+1)\,q]$th moments of the current distribution in the nonnoisy network[5,10-12]

$$M_q \simeq \sum_{\text{bonds}} i^{(\beta+1)q} \sim L^{\tilde{\psi}(q,\beta)} \tag{3}$$

thus offering a connection between the two distributions. We will assume boundary conditions of a unit current injected into one terminal and extracted from the other. It was recently shown that the power law behavior of M_q breaks down for values of q that are smaller than some small negative value $q_c < 0$.[8] Therefore we will concentrate on positive values of q and disregard the nonmultifractal behavior for $q < 0$ altogether.

Much information has been accumulated about nonlinear RRNs.[9,10] In particular, the value of $\tilde{\psi}(q, \beta)$ is known exactly for several values of q and β and is known with great accuracy for two lines in the β–q plane: (i) $\beta = 1$ and all q,[5,8] and (ii) $q = 1$ and all β.[10] The exact values are listed in Table I. Generally, $\tilde{\psi}(q, \beta)$ is a nonincreasing function of q and of β. Moreover, it is convex as a function of q for all β except at $\beta = -1, 0^-, 0^+$, and ∞, where $\partial\tilde{\psi}(q, \beta)/\partial\beta$ vanishes for all q.[10]

To map from $\tilde{\psi}(q, \beta)$ to $f(\alpha, \beta)$, first note that $M_1(\beta) = R(\beta)$ in (2) is the nonlinear resistance. Therefore, defining a "bond probability"

$$p_b = \frac{i_b^{\beta+1}}{R(\beta)} \tag{4}$$

we obtain a proper normalized distribution $\sum_b p_b = 1$ whose qth moment is

$$\mu_q \equiv \sum_b p_b^q = \frac{M_q(\beta)}{R^q(\beta)} \sim L^{\tilde{\psi}(q,\beta) - q\tilde{\zeta}(\beta)} \tag{5}$$

Table I. Exact Values of $\tilde{\psi}(q, \beta)$ for Several Values of q and β[a]

$\tilde{\psi}(0, \beta) = D_B$
$\tilde{\psi}(1, 1) = \tilde{\zeta}_R$
$\tilde{\psi}(\infty, \beta) = 1/\nu$
$\tilde{\psi}(q, \infty) = 1/\nu$
$\tilde{\psi}(q, 0^+) = \tilde{\zeta}_{\min}$
$\tilde{\psi}(q, 0^-) = \tilde{\zeta}_{\max}$
$\tilde{\psi}(q, -1) = D_B$
$\tilde{\psi}(q, -\infty) = -\beta z$

[a] $\tilde{\zeta}_{\min}$ is the critical exponent of the minimal path between the terminals; $\tilde{\zeta}_{\max}$ is the critical exponent of the maximal path between the terminals; $z = \log N_0/\log L$, where N_0 is the maximal number of bonds that one can cut by a simply connected surface so as to render the two terminals disconnected.

where $\tilde{\zeta}(\beta) = \tilde{\psi}(1, \beta)$. Now we can follow Halsey *et al.*[3] and define a set of fractal dimensionalities $D(q)$ via

$$\mu_q \sim L^{(1-q)D(q)}, \qquad D(q) = \frac{q\tilde{\zeta}(\beta) - \tilde{\psi}(q, \beta)}{q-1} \tag{6}$$

It is easy to see that $D(0)$ is the fractal dimension of the backbone D_B, $D(\infty) = \tilde{\zeta}(\beta)$, and $D(1) = \tilde{\zeta}(\beta) - \partial/\partial q[\tilde{\psi}(q, \beta)]_{q=1}$. In ref. 7 it was shown for the linear network that

$$\alpha(q) = \tilde{\psi}(1) - \frac{d}{dq}\tilde{\psi}(q) \tag{7a}$$

$$f(q) = \tilde{\psi}(q) - q\frac{d}{dq}\tilde{\psi}(q) \tag{7b}$$

These relations hold for the nonlinear case as well, where the exponents $\tilde{\psi}(q)$ become β dependent and the derivatives are partial at constant β. Eliminating q between Eqs. (7), one obtains the nonlinear multifractal function $f(\alpha, \beta)$. Thus, relations (7) constitute the mapping between $\tilde{\psi}(q, \beta)$ and $f(\alpha, \beta)$. The aforementioned restriction on the positivity of q limits α and f to the ranges

$$\begin{aligned} &\tilde{\zeta}(\beta) \leqslant \alpha \leqslant \alpha_{\max} = \tilde{\zeta}(\beta) - \left(\frac{\partial\tilde{\psi}(q, \beta)}{\partial q}\right)_{q=0} \\ &1/\nu \leqslant f \leqslant D_B \end{aligned} \tag{8}$$

Next we write $f(\alpha, \beta)$ in terms of another variable to find general relations that will be used below. We scale α by

$$\omega = \frac{\alpha - \alpha_{\min}}{\alpha_{\max} - \alpha_{\min}} = \frac{\tilde{\zeta}(\beta) - \alpha}{(\partial\tilde{\psi}/\partial q)_{q=0}} = \frac{(\partial\tilde{\psi}/\partial q)}{(\partial\tilde{\psi}/\partial q)_{q=0}} \tag{9}$$

where we inserted the extreme values of α that correspond to $q = 0, \infty$, respectively. Using relations (7), we can obtain a general expression for the multifractal function in terms of the new variable,

$$f(\omega, \beta) = \left(\frac{\partial\tilde{\psi}}{\partial q}\right)_{q=0}\left[\int_0^q \omega(q')\, dq' - q\omega\right] + D_B \tag{10}$$

where the constant of integration is determined by the condition $f(q=0, \beta) = D_B$. Generally there may appear a dependence of $f(\omega, \beta)$ on β via the fact that for constant ω, q is a function of β. However, approximating $\tilde{\psi}(q, \beta)$, as we do below, and mapping the approximation to $f(\omega, \beta)$, we find that the multifractal function becomes independent of β.

3. AN AF FOR $f(\alpha, \beta)$

Next we construct an AF for $\tilde{\psi}(q, \beta)$ by adopting the exponential form proposed for the linear $\tilde{\psi}(q, 1)$ in ref. 8. Replace the linear conductivity exponent $\tilde{\zeta}_R$ by $\tilde{\zeta}(\beta)$. Then our approximation is

$$\tilde{\psi}(q, \beta) = 1/v + (D_B - 1/v)\, e^{q\gamma(\beta)} \tag{11}$$

where

$$\gamma(\beta) = \ln \frac{\tilde{\zeta}(\beta) - 1/v}{D_B - 1/v}$$

For the value of $\tilde{\zeta}(\beta)$, let us use an approximation proposed in ref. 10. Equation (11) provides a good description for real percolative systems whenever $\beta \gtrsim 1$. In the past it was shown that a renormalization on the

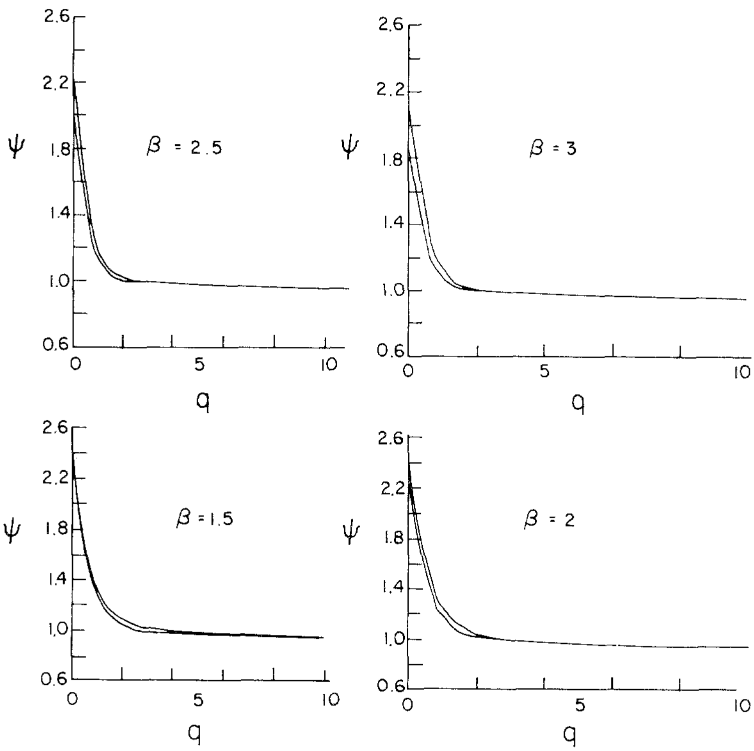

Fig. 1. The function $\tilde{\psi}(q, \beta)$ for several values of β: the lower line consists of the approximant exponential function, while the upper line comprises an exact solution on the Mandelbrot–Given structure.

Mandelbrot–Given structure yields good estimation for many critical exponents for linear two-dimensional resistor networks.[13] Therefore, comparing between our approximation and the noise exponents (that can be found exactly) on this structure, one can assess the adequacy of the AF. As we see from Fig. 1, the two curves look very close for different values of β. As β decreases below 1, discrepancies appear for larger and larger values of q, rendering (11) inapplicable for too small values of β. These discrepancies may arise from the fact that we actually compare the AF with another approximation rather than with real data. Another source for error may be the fact that we approximate the contribution of the blobs' resistance to the power $1/v$ (caused by the singly connected bonds) with a single exponential form, rather than by many such terms, as occurs in real systems. Nevertheless, by limiting this analysis for $\beta \gtrsim 1$, one can regard (11) as a good description. For $\beta = 1$, (11) was found to describe accurately real percolative networks (see Fig. 1 in ref. 8). Using (7) and eliminating q, we find

$$f(\alpha, \beta) = 1/v + \Phi(\alpha, \beta)\left(1 + \ln \frac{D_B - 1/v}{\Phi(\alpha, \beta)}\right) \tag{12}$$

where $\Phi(\alpha, \beta) = [\tilde{\zeta}(\beta) - \alpha]/\gamma(\beta)$. Thus, we have an explicit approximation for the multifractal function in nonlinear networks.

4. UNIVERSAL AND SUPERUNIVERSAL MULTIFRACTAL FUNCTIONS

I proceed to show that the above approximation displays a universal behavior for all values of β to which it is valid. Applying (10) for the approximant function (12), let us identify $\gamma(\beta) = \ln(\omega)/q$ and $\Phi(\omega, \beta) = \omega(D_B - 1/v)$, which yields

$$f(\omega, \beta) = 1/v + \omega(D_B - 1/v)(1 - \ln \omega) \tag{13}$$

Expression (14) is independent of β for fixed ω, namely, when plotting the multifractal function vs. ω for any value of β, all the lines will collapse on a universal curve that is independent of β.

As a check on this conjectured universality, I plot $f(\omega)$ for several values of β (taking a very wide range of values) for the Mandelbrot–Given structure. The resulting curves do appear to collapse to a nearly universal line for all values of β greater then $\beta \gtrsim 1$, as is shown in Fig. 2. I discuss the ramifications of this universality below.

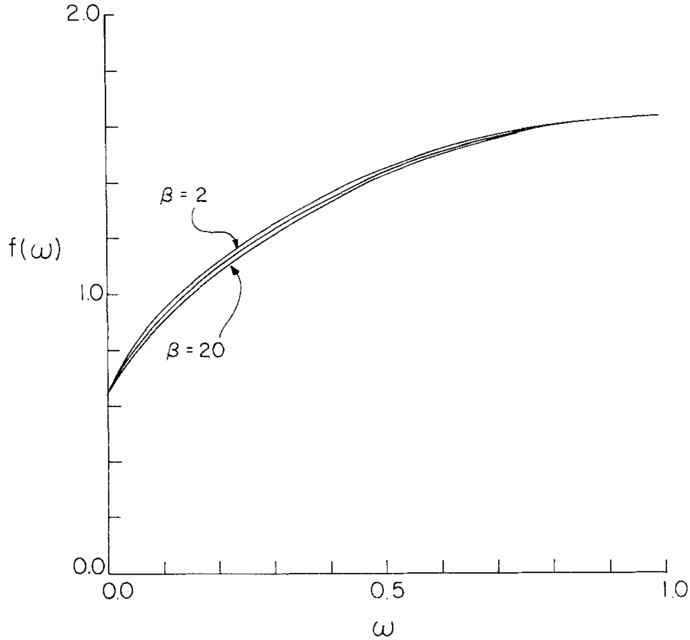

Fig. 2. Plot of $f(\omega, \beta)$ vs. ω for $\beta = 2.0$, 3.0, and 20.

Inspecting (13), one realize that it depends on the dimension d only through the exponents D_B and ν. However, if one scales now $f(\omega)$ by

$$g(\omega) = \frac{f - 1/\nu}{D_B - 1/\nu} \tag{14}$$

one obtains a more striking result. The function $g(\omega)$ depends only on ω and is superuniversal with respect to the dimension,

$$g(\omega) = \omega(1 - \ln \omega) \tag{15}$$

Thus, this information yields a function that is valid for all dimensions. Due to the good numerical results that the approximation for $\tilde{\psi}(q, \beta)$ yields in two dimensions, it seems plausible that if not exactly superuniversal, then this description is very close to being such in real percolating RRNs. I elaborate on this point in the following concluding section.

5. CONCLUSION AND DISCUSSION

To conclude, I have generalized the multifractal formalism to describe nonlinear random resistor networks. I constructed an explicit approximant function for the resulting multifractal function $f(\alpha, \beta)$ that mimicks well real nonlinear percolative RRNs, though it is not exact.[8] Next I showed that $f(\alpha, \beta)$ can be made universal by scaling α to a new variable. Employing the accuracy of the approximation for $\beta \gtrsim 1$, I conjecture that this universality is "almost" exact for real RRNs. Such a universality is of practical importance because (i) it was shown that many topological properties of the percolative underlying network can be derived by studying the nonlinear conductivity problem,[9,10] (ii) another ramification of this conjecture concerns the study of noise in low-temperature ceramics that display a power law characteristic $V-I$ function.[14] Since the power in such functions is temperature dependent, then universality in this case implies temperature independence in the right variables. Hence, I predict that, in probing the shape of the function $f(\omega)$ for powers β that are larger than one, almost no change will be detected as a function of the temperature.

Next I transformed the approximate $f(\omega)$ to a superuniversal dimension independent function, $g(\omega)$. To the best of my knowledge, no tests of this property were done in the past for real percolating RRNs. Nevertheless, there exist accurate numerical results for $\tilde{\psi}(q, 1)$ in two and three dimensions.[5,8] Therefore, one can check the superuniversality by transforming to $g(\omega)$ at $\beta = 1$ and comparing the two curves. However, one should bear in mind that $\beta \simeq 1$ may be too near to the lower limit for the conjecture to hold. Another check can be performed in $6-\varepsilon$ dimensions where the functional dependence of the function $\tilde{\psi}(q, \beta)$ on the dimensionality is exactly known.[15] These two checks are under way and will be reported in a forthcoming paper. One can use these properties of $g(\omega)$ to measure the behavior in low-dimensional systems for inferring the behavior in higher dimensions. Alternatively, one can use the universality to measure properties of an RRN with a given nonlinearity to conclude for a system having a different one. Moreover, if it can be shown that linear networks can also be described adequately by this universality, then one can actually probe properties of nonlinear networks by measuring linear ones. Thus, the universality and the superuniversality of $g(\omega)$ may turn out to be a very useful tool in the study of noisy nonlinear networks.

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